Semi-analytical Proof of Abelian Dominance on Confinement in the Maximally Abelian Gauge

H. Ichie*
Department of Physics, Tokyo Institute of Technology
Meguro, Tokyo 152-8551, Japan

H. Suganuma†
Research Center for Nuclear Physics (RCNP), Osaka University
Ibaraki, Osaka 567-0047, Japan

Abstract

We study abelian dominance for confinement in terms of the local gluon properties in the maximally abelian (MA) gauge in a semi-analytical manner with the help of the lattice QCD. The global Weyl symmetry persistently remains as the relic of SU(N_c) in the MA gauge, and provides the ambiguity on the electric and magnetic charges. We derive the criterion on the SU(N_c)-gauge invariance in terms of the residual symmetry in the abelian gauge. In the lattice QCD, we find microscopic abelian dominance on the link variable for the whole region of β in the MA gauge. The off-diagonal angle variable, which is not constrained by the MA-gauge fixing condition, tends to be random besides the residual gauge degrees of freedom. Within the random-variable approximation for the off-diagonal angle variable, we prove that off-diagonal gluon contribution to the Wilson loop obeys the perimeter law in the MA gauge, and show exact abelian dominance for the string tension, although small deviation is brought by the finite size effect of the Wilson loop in the actual lattice QCD simulation.

*E-mail: ichie@th.phys.titech.ac.jp

†E-mail: suganuma@rcnp.osaka-u.ac.jp

I. INTRODUCTION

Quantum Chromodynamics (QCD) is the fundamental theory of the strong interaction [1,2]. Due to the asymptotic freedom, the gauge-coupling constant of QCD becomes small in the high-energy region and the perturbative QCD provides a direct and systematic description of the QCD system in terms of quarks and gluons. On the other hand, in the low-energy region, the strong gauge-coupling nature of QCD leads to nonperturbative features like color confinement, dynamical chiral-symmetry breaking [3–5], and nontrivial topological effect by instantons [6–8], and it is impossible to understand them directly from quarks and gluons in a perturbative manner. Instead of quarks and gluons, some collective or composite modes may be relevant degrees of freedom for the nonperturbative description in the infrared region of QCD. As for chiral dynamics, the pion and the sigma meson play the important role for the low-energy QCD, and they are included in the QCD effective theories like the (non-) linear sigma model, the chiral bag model [9,10] and the Nambu-Jona-Lasinio model [3,11], where these mesons are described as composite modes of quarks and anti-quarks. Here, the pion is considered to be the Nambu-Goldstone boson relating to spontaneous chiral-symmetry breaking and obeys the low-energy theorem [12] and the current algebra [1]. On the other hand, confinement is essentially described by dynamics of gluons rather than quarks. Hence, it is desired to extract the relevant collective mode from gluons for confinement phenomena.

In 1970's, Nambu, 't Hooft and Mandelstam proposed an interesting idea that quark confinement can be interpreted using the dual version of the superconductivity [13–15]. In the ordinary superconductor, Cooper-pair condensation leads to the Meissner effect, and the magnetic flux is excluded or squeezed like a quasi-one-dimensional tube as the Abrikosov vortex [16], where the magnetic flux is quantized topologically. On the other hand, from the Regge trajectory [17] of hadrons and the lattice QCD simulation [18,19], the confinement force between the color-electric charge is characterized by the universal physical quantity of the string tension, and is brought by one-dimensional squeezing of the color-electric flux [20] in the QCD vacuum. Hence, the QCD vacuum would be regarded as the dual version of the superconductor based on above similarities on the low-dimensionalization of the quantized flux between charges. In this dual-superconductor picture for the QCD vacuum, the squeezing of the color-electric flux between quarks is realized by the dual Meissner effect as the result of condensation of color-magnetic monopoles, which is the dual version of electric-charge condensation. Monopole condensation and its relevant role for confinement were analytically pointed out by Seiberg and Witten very recently in the $N = 2$ supersymmetric version of QCD [21].

However, there are two following large gaps between QCD and the dual superconductor picture.

1. This picture is based on the abelian gauge theory subject to the Maxwell-type equations, where electro-magnetic duality is manifest, while QCD is a nonabelian gauge theory.

2. The dual-superconductor scenario requires condensation of (color-)magnetic monopoles as the key concept, while QCD does not have such a monopole as the elementary degrees of freedom.

As the connection between QCD and the dual superconductor scenario, 't Hooft proposed the concept of the abelian gauge fixing [22], the partial gauge fixing which only remains abelian gauge degrees of freedom in QCD. By definition, the abelian gauge fixing reduces QCD into an abelian gauge theory, where the off-diagonal element of the gluon field behaves as a charged matter field and provides a color-electric current in terms of the residual abelian gauge symmetry. As a remarkable fact in the abelian gauge, color-magnetic monopoles appear as topological objects corresponding to the nontrivial homotopy group $\Pi_2(\mathrm{SU}(N_c)/\mathrm{U}(1)^{N_c-1}) = \mathbf{Z}_\infty^{N_c-1}$. Thus, by the abelian gauge fixing, QCD is reduced into an abelian gauge theory including both the electric current j_μ and the magnetic current k_μ, which is expected to provide the theoretical basis of the monopole-condensation scheme for the confinement mechanism.

As for the irrelevance of the off-diagonal gluon gluons, Ezawa and Iwazaki assumed abelian dominance that the only abelian gauge fields with monopoles would be essential for the description of nonperturbative phenomena in the low-energy region of QCD, and showed a possibility of monopole condensation in the infrared scale by investigating "energy-entropy balance" on the monopole current [23–26] in a similar way to the Kosterliz-Thouless transition in the 1+2 dimensional superconductivity [27]. Ezawa and Iwazaki formulated the dual London theory as an infrared effective theory of QCD, and later it is reformulated as the dual Ginzburg-Landau theory [28–34].

Furthermore, abelian dominance [35–39] and monopole condensation [40–42] have been investigated using the lattice QCD simulation in the maximally abelian (MA) gauge [40–43]. The MA gauge is the abelian gauge where the diagonal component of the gluon is maximized by the gauge transformation. In the MA gauge, physical information of the gauge configuration is concentrated into the diagonal component as much as possible. The lattice QCD studies indicate *abelian dominance* that the string tension [35–37] and the chiral condensate [38,39] are almost described only by abelian variables in the MA gauge. Moreover, *monopole dominance* is also observed in the lattice QCD simulation in the MA gauge: only the monopole part in the abelian variable contributes to the nonperturbative QCD [37,38]. Thus, the lattice QCD phenomenology suggests the dominant role of abelian variables including monopoles for the nonperturbative QCD in the MA gauge [44,45].

In this paper, we aim to understand the origin of abelian dominance for confinement

in terms of the local gluon properties in the MA gauge, and study the relation between macroscopic abelian dominance and microscopic abelian dominance in a semi-analytical manner with the help of the lattice QCD simulation. In Section 2, we study the residual symmetry and the gauge invariance condition for operators in the 't Hooft abelian gauge, paying the attention to the global Weyl symmetry. In Section 3, we investigate the MA gauge both in the lattice and in the continuum theories in terms of the gauge connection. In Section 4, we introduce the abelian projection rate as the overlapping factor between SU(2) and abelian link variables, and study microscopic abelian dominance in the MA gauge in the lattice formalism. In Section 5, we study the contribution of off-diagonal gluons to the Wilson loop in the MA gauge, and prove abelian dominance for the string tension in a semi-analytical manner. Section 6 is devoted to summary and concluding remarks.

II. RESIDUAL SYMMETRY AND GAUGE INVARIANCE CONDITION IN THE ABELIAN GAUGE

The dual superconductor picture for confinement phenomena is based on the abelian gauge theory including monopoles, and the 't Hooft abelian gauge fixing [22] is the key concept for the connection from QCD to such an abelian gauge theory. In this section, we investigate the abelian gauge fixing in QCD in terms of the residual gauge symmetry.

The abelian gauge fixing, the partial gauge fixing which remains the abelian gauge symmetry, is realized by the diagonalization of a suitable SU(N_c)-gauge dependent variable as $\Phi[A_\mu(x)] \in su(N_c)$ by the SU(N_c) gauge transformation. In the abelian gauge, $\Phi[A_\mu(x)]$ plays the similar role of the Higgs field, and can be regarded as the composite Higgs field.

For an hermite operator $\Phi[A_\mu(x)]$ which obeys the adjoint transformation, $\Phi(x)$ is transformed as

$$\begin{aligned}\Phi(x) = \Phi^a T^a \rightarrow \Phi^\Omega(x) = \Omega(x)\Phi(x)\Omega^\dagger(x) &\equiv \vec{H}\cdot\vec{\Phi}_{diag}(x) \\ &= \mathrm{diag}(\lambda^1(x), \cdots, \lambda^{N_c}(x)), \end{aligned} \tag{1}$$

using a suitable gauge function $\Omega(x) = \exp\{i\xi^a(x)T^a\} \in \mathrm{SU}(N_c)$. Here, each diagonal component λ^i (i=1,$\cdots$,N_c) is to be real for the hermite operator $\Phi[A_\mu(x)]$. In the abelian gauge, the SU(N_c) gauge symmetry is reduced into the $\mathrm{U}(1)^{N_c-1}$ gauge symmetry corresponding to the gauge-fixing ambiguity. The operator $\Phi(x)$ is diagonalized to $\vec{H}\cdot\vec{\Phi}_{diag}(x)$ also by the gauge function $\Omega^\omega(x) \equiv \omega(x)\Omega(x)$ with $\omega(x) = \exp(-i\,\vec{H}\cdot\vec{\varphi}(x)) \in \mathrm{U}(1)^{N_c-1}$,

$$\Phi(x) \rightarrow \Omega^\omega(x)\Phi(x)\Omega^{\omega\dagger}(x) = \omega(x)\vec{H}\cdot\vec{\Phi}_{diag}(x)\omega^\dagger(x) = \vec{H}\cdot\vec{\Phi}_{diag}(x), \tag{2}$$

and therefore $\mathrm{U}(1)^{N_c-1}$ abelian gauge symmetry remains in the abelian gauge.

In the abelian gauge, there also remains the global Weyl symmetry as a "relic" of the nonabelian theory [46,47]. Here, the Weyl symmetry corresponds to the subgroup of $\mathrm{SU}(N_c)$ relating to the permutation of the basis in the fundamental representation. Then, the Weyl group is expressed as the permutation group $\mathbf{P}_{N_c}$ including ${}_{N_c}C_2$ elements. For simplicity, let us consider the $N_c = 2$ case. For SU(2) QCD, the Weyl symmetry corresponds to the interchange of the SU(2)-quark color, $|+\rangle \equiv \binom{1}{0}$ and $|-\rangle \equiv \binom{0}{1}$, in the fundamental representation. The global Weyl transformation is expressed by the global gauge function,

$$
\begin{aligned}
W &= e^{i\{\frac{\tau_1}{2}\cos\alpha + \frac{\tau_2}{2}\sin\alpha\}\pi} = i\{\tau_1\cos\alpha + \tau_2\sin\alpha\} \\
&= i\begin{pmatrix} 0 & e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix} \in \mathbf{P}_2 \subset \mathrm{SU}(2)
\end{aligned}
\tag{3}
$$

with an arbitrary constant $\alpha \in \mathbf{R}$. By the global Weyl transformation W, the SU(2)-quark color is interchanged as $W|+\rangle = ie^{i\alpha}|-\rangle$ and $W|-\rangle = ie^{-i\alpha}|+\rangle$ except for the global phase factor. This global Weyl symmetry remains in the abelian gauge, because the operator $\Phi(x)$ is also diagonalized by using $\Omega^W(x) \equiv W\Omega(x)$,

$$
\Phi(x) \to \Omega^W(x)\Phi(x)\Omega^{W\dagger}(x) = W\Phi_{diag}(x)\frac{\tau_3}{2}W^\dagger = -\Phi_{diag}(x)\frac{\tau_3}{2},
\tag{4}
$$

Here, the sign of $\Phi_{diag}(x)$, or the order of the diagonal component $\lambda^i(x)$, is globally changed by the Weyl transformation. It is noted that the sign of the $\mathrm{U}(1)_3$ gauge field $\mathcal{A}_\mu \equiv A_\mu^3 \frac{\tau_3}{2}$ is globally changed under the Weyl transformation,

$$
\mathcal{A}_\mu \to \mathcal{A}_\mu^W = WA_\mu^3\frac{\tau_3}{2}W^\dagger = -A_\mu^3\frac{\tau_3}{2} = -\mathcal{A}_\mu.
\tag{5}
$$

Therefore, all the sign of the abelian field strength, electric and magnetic charges are also globally changed:

$$
\begin{aligned}
&\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu}\frac{\tau_3}{2} \to \mathcal{F}_{\mu\nu}^W = W\mathcal{F}_{\mu\nu}W^\dagger = -\mathcal{F}_{\mu\nu}, \\
&j_\mu \equiv \partial^\alpha \mathcal{F}_{\alpha\mu} \to j_\mu^W = -j_\mu, \\
&k_\mu \equiv \partial^{\alpha *}\mathcal{F}_{\alpha\mu} \to k_\mu^W = -k_\mu.
\end{aligned}
\tag{6}
$$

In the abelian gauge, the absolute signs of the electric and the magnetic charges are settled, only when the Weyl symmetry is fixed by the additional condition. When $\Phi[A_\mu(x)]$ obeys the adjoint-type gauge transformation like the nonabelian Higgs field, the global Weyl symmetry can be easily fixed by imposing the additional gauge-fixing condition as $\Phi_{diag}(x) \ge 0$ for SU(2), or the ordering condition of the diagonal components λ^i in $\vec{H}\cdot\vec{\Phi}_{diag}$ as $\lambda^1 \ge .. \ge \lambda^{N_c}$ for the $\mathrm{SU}(N_c)$ case. As for the appearance of monopoles in the abelian

gauge, the global Weyl symmetry $\mathbf{P}_{N_c}$ is not relevant, because the nontriviality of the homotopy group is not affected by the global Weyl symmetry. However, the definition of the magnetic monopole charge, which is expressed by the nontrivial dual root of $\mathrm{SU}(N_c)_{\mathrm{dual}}$ [23], is globally changed by the Weyl transformation.

Now, we consider the abelian gauge fixing in terms of the coset space of the fixed gauge symmetry. The abelian gauge fixing is a sort of the partial gauge fixing which reduces the gauge group $G \equiv \mathrm{SU}(N_c)_{\mathrm{local}}$ of the system into its subgroup $H \equiv \mathrm{U}(1)^{N_c-1}_{\mathrm{local}}(\times \mathrm{P}^{\mathrm{global}}_{N_c})$ including the maximally torus subgroup of G. In other words, the abelian gauge fixing freezes the gauge symmetry relating to the coset space G/H, and hence the representative gauge function Ω which brings the abelian gauge belongs the coset space G/H: $\Omega \in G/H$. In fact, $\Omega \in G/H$ is uniquely determined without the ambiguity on the residual symmetry H, if the additional condition on H is imposed for Ω.

However, such a partial gauge fixing makes the total gauge invariance unclear. Here, let us consider the $\mathrm{SU}(N_c)$ gauge-invariance condition on the operator defined in the abelian gauge [46]. To begin with, we investigate the gauge-transformation property of the gauge function $\Omega \in G/H$ which brings the abelian gauge (See. Fig.1). For simplicity, the operator Φ to be diagonalized is assumed to obey the adjoint gauge transformation as $\Phi \rightarrow \Phi^g = g\Phi g^\dagger$ with $^\forall g \in G$. After the gauge transformation by $^\forall g \in G$, $\Omega^g \in G/H$ is defined so as to diagonalize Φ^g as $\Omega^g \Phi^g (\Omega^g)^\dagger = \Phi_{diag}$, and hence the gauge function $\Omega^g \in G/H$ which realizes the abelian gauge is transformed as

$$\Omega \rightarrow \Omega^g = h[g]\Omega g^\dagger \tag{7}$$

under arbitrary $\mathrm{SU}(N_c)$ gauge transformation by $g \in G$. Here, $h[g] \in H$ is chosen so as to make Ω^g belong G/H, i.e., $\Omega^g \in G/H$. (If the additional condition on H is imposed to specify $\Omega \in G/H$, $\Omega g^\dagger$ does not satisfy it in general.) This is similar to the argument on the hidden local symmetry [12] in the nonlinear representation. In general, the gauge function $\Omega \in G/H$ is transformed nonlinearly by the gauge function g due to $h[g] \in H$. Thus, the gauge-transformation property on the gauge function $\Omega \in G/H$ becomes nontrivial in the partial gauge fixing.

Owing to the nontrivial transformation (7) of $\Omega \in G/H$, any operator O^Ω defined in the abelian gauge is found to be transformed as $O^\Omega \rightarrow (O^\Omega)^{h[g]}$ by the $\mathrm{SU}(N_c)$ gauge transformation of $^\forall g \in G$. We demonstrate this for the gluon field $A_\mu^\Omega \equiv \Omega(A_\mu + \frac{i}{e}\partial_\mu)\Omega^\dagger$ in the abelian gauge. By the gauge transformation of $^\forall g \in G$, A_μ^Ω is transformed as

$$A_\mu^\Omega \rightarrow (A_\mu^g)^{\Omega^g} = A_\mu^{\Omega^g g} = A_\mu^{h[g]\Omega} = (A_\mu^\Omega)^{h[g]} = h[g](A_\mu^\Omega + \frac{i}{e}\partial_\mu)h^\dagger[g]. \tag{8}$$

Here, we have used

$$(A_\mu^{g_1})^{g_2} = g_2(A_\mu^{g_1} + \frac{i}{e}\partial_\mu)g_2^\dagger = (g_2 g_1)(A_\mu + \frac{i}{e}\partial_\mu)(g_2 g_1)^\dagger = (A_\mu)^{g_2 g_1} \tag{9}$$

for the successive gauge transformation by $g_1,\ g_2 \in \mathrm{SU}(N_c)$. Similarly, the operator O^Ω defined in the abelian gauge is transformed by ${}^\forall g \in G$ as

$$\begin{aligned} O^\Omega \to (O^g)^{\Omega^g} &= \Omega^g O^g \Omega^{g\dagger} = h[g]\Omega g^\dagger \cdot g O g^\dagger \cdot g \Omega^\dagger h^\dagger[g] \\ &= h[g]\Omega O \Omega^\dagger h^\dagger[g] = h[g] O^\Omega h^\dagger[g] = (O^\Omega)^{h[g]}, \end{aligned} \tag{10}$$

as shown in Fig.1. Here, O is assumed to obey the adjoint transformation as $O^g = gOg^\dagger$ for simplicity.

Thus, arbitrary $\mathrm{SU}(N_c)$ gauge transformation by $g \in G$ is mapped into the partial gauge transformation by $h[g] \in H$ for the operator O^Ω defined in the abelian gauge, and O^Ω transforms nonlinearly as $O^\Omega \to (O^\Omega)^{h[g]}$ by the $\mathrm{SU}(N_c)$ gauge transformation g. If the operator O^Ω is H-invariant, one gets $(O^\Omega)^{h[g]} = O^\Omega$ for any $h[g] \in H$, and hence O^Ω is also G-invariant or total $\mathrm{SU}(N_c)$ gauge invariant, because O^Ω is transformed into $(O^\Omega)^{h[g]} = O^\Omega$ by ${}^\forall g \in G$. Thus, we find a useful criterion on the $\mathrm{SU}(N_c)$ gauge invariance of the operator defined in the abelian gauge [46]: If the operator O^Ω defined in the abelian gauge is H-invariant, O^Ω is also invariant under the whole gauge transformation of G.

Here, let us consider the application of this criterion to the effective theory of QCD in the abelian gauge, the dual Ginzburg-Landau (DGL) theory [28,29]. In the DGL theory, the local $\mathrm{U}(1)^{N_c-1}$ and the global Weyl symmetries remain, and the dual gauge field B_μ and the monopole field χ_α $[\alpha=1,\ \cdots,\ \frac{1}{2}N_c(N_c-1)]$ are the relevant modes for infrared physics. Although B_μ is invariant under the local transformation of $\mathrm{U}(1)^{N_c-1} \subset \mathrm{SU}(N_c)$, $B_\mu \equiv \vec{B}_\mu \cdot \vec{H}$ is variant under the global Weyl transformation, and therefore B_μ is $\mathrm{SU}(N_c)$-gauge dependent object and does not appear in the real world alone. As for the monopole field, there exists one Weyl-invariant combination of the monopole field fluctuation, $\tilde{\chi} \equiv \sum_\alpha \tilde{\chi}_\alpha$ [32], which is also $\mathrm{U}(1)^{N_c-1}$-invariant. Therefore, the monopole fluctuation $\tilde{\chi}$ is completely residual-gauge invariant in the abelian gauge, so that $\tilde{\chi}$ is $\mathrm{SU}(N_c)$-gauge invariant and is expected to appear as a scalar glueball with $J^{PC} = 0^{++}$, like the Higgs particle in the standard model.

III. MAXIMALLY ABELIAN (MA) GAUGE IN THE CONNECTION FORMALISM

The abelian gauge has some arbitrariness corresponding to the choice of the operator Φ to be diagonalized. As the typical abelian gauge, the maximally abelian (MA) gauge, the Polyakov gauge and the F12 gauge have been tested on the dual superconductor scenario for the nonperturbative QCD. Recent lattice QCD studies show that infrared phenomena such as confinement properties and chiral symmetry breaking are almost reproduced in the MA gauge [35–41]. In the SU(2) lattice formalism, the MA gauge is defined so as to maximize

$$
\begin{aligned}
R_{\rm MA}[U_\mu] &\equiv \sum_{s,\mu} {\rm tr}\{U_\mu(s)\tau_3 U_\mu^\dagger(s)\tau_3\} \\
&= 2\sum_{s,\mu}\{U_\mu^0(s)^2 + U_\mu^3(s)^2 - U_\mu^1(s)^2 - U_\mu^2(s)^2\} \\
&= 2\sum_{s,\mu}\left[1 - 2\{U_\mu^1(s)^2 + U_\mu^2(s)^2\}\right]
\end{aligned} \tag{11}
$$

by the SU(2) gauge transformation. Here, the link variable $U_\mu(s) \equiv U_\mu^0(s) + i\tau^a U_\mu^a(s) \in$ SU(2) with $U_\mu^0(s)$, $U_\mu^a(s) \in \mathbf{R}$ relates to the (continuum) gluon field $A_\mu \equiv A_\mu^a T^a \in su(2)$ as $U_\mu(s) = e^{iaeA_\mu(s)}$, where e denotes the QCD gauge coupling and a the lattice spacing. In the MA gauge, the absolute value of off-diagonal components, $U_\mu^1(s)$ and $U_\mu^2(s)$, are forced to be small. In the continuum limit $a \to 0$, the link variable reads $U_\mu(s) = e^{iaeA_\mu(s)} = 1 + iaeA_\mu(s) + O(a^2)$, and hence the MA gauge is found to minimize the functional,

$$
R_{ch}[A_\mu] \equiv \frac{1}{2}e^2 \int d^4x \{A_\mu^1(x)^2 + A_\mu^2(x)^2\} = e^2 \int d^4x A_\mu^+(x) A_\mu^-(x), \tag{12}
$$

with $A_\mu^\pm(x) \equiv \frac{1}{\sqrt{2}}\{A_\mu^1(x) \pm iA_\mu^2(x)\}$. Thus, in the MA gauge, the off-diagonal gluon component is globally forced to be small by the gauge transformation, and hence the QCD system is expected to be describable only by its diagonal part approximately.

The MA gauge is a sort of the abelian gauge which diagonalizes the hermite operator

$$
\Phi[U_\mu(s)] \equiv \sum_{\mu,\pm} U_{\pm\mu}(s)\tau_3 U_{\pm\mu}^\dagger(s). \tag{13}
$$

Here, we use the convenient notation $U_{-\mu}(s) \equiv U_\mu^\dagger(s-\hat\mu)$ in this paper. In the continuum limit, the condition of the MA gauge becomes $\sum_\mu (i\partial_\mu \pm eA_\mu^3)A_\mu^\pm = 0$. This condition can be regarded as the maximal decoupling condition between the abelian gauge sector and the charged gluon sector.

In the MA gauge, $\Phi(s)$ is diagonalized as $\Phi_{\rm MA}(s) = \Phi_{diag}(s)\tau_3$ with $\Phi_{diag}(s) \in \mathbf{R}$, and there remain the local $\mathrm{U}(1)_3$ symmetry and the global Weyl symmetry [46]. As a remarkable fact, $\Phi(s)$ does not obey the adjoint transformation in the MA gauge, and the sign of $\Phi_{diag}(s)$ is not changed by the Weyl transformation by W in Eq.(3),

$$
\begin{aligned}
&\Phi_{\rm MA}(s) = \Phi_{diag}(s)\tau_3 \\
&\to \Phi_{\rm MA}^W(s) = \sum_{\mu,\pm} W U_{\pm\mu}(s) W^\dagger \tau_3 W U_{\pm\mu}^\dagger(s) W^\dagger \\
&\quad = -\sum_{\mu,\pm} W U_{\pm\mu}(s)\tau_3 U_{\pm\mu}^\dagger(s) W^\dagger = -W\Phi_{diag}(s)\tau_3 W^\dagger = \Phi_{diag}(s)\tau_3.
\end{aligned} \tag{14}
$$

Thus, the Weyl symmetry is not fixed in the MA gauge by the simple ordering condition as $\Phi_{diag} \geq 0$, unlike the adjoint case. We find the gauge invariance condition on the

operator O^Ω defined in the MA gauge: if O^Ω is invariant both under the local $\mathrm{U}(1)^{N_c-1}$ gauge transformation and the global Weyl transformation, O^Ω is also invariant under the $\mathrm{SU}(N_c)$ gauge transformation.

In the continuum $\mathrm{SU}(N_c)$ QCD, it is more fundamental and convenient to define the MA gauge fixing by way of the $\mathrm{SU}(N_c)$-covariant derivative operator $\hat{D}_\mu \equiv \hat{\partial}_\mu + ieA_\mu$, where $\hat{\partial}_\mu$ is the derivative operator satisfying $[\hat{\partial}_\mu, f(x)] = \partial_\mu f(x)$. The MA gauge is defined so as to make $\mathrm{SU}(N_c)$-gauge connection $\hat{D}_\mu = \hat{\partial}_\mu + ieA_\mu^a T^a$ close to $\mathrm{U}(1)^{N_c-1}$-gauge connection $\hat{D}_\mu^{\rm Abel} = \hat{\partial}_\mu + ie\vec{A}_\mu \cdot \vec{H}$ by minimizing

$$
\begin{aligned}
R_{\rm ch} &\equiv \int d^4x \, {\rm tr}[\hat{D}_\mu, \vec{H}]^\dagger [\hat{D}_\mu, \vec{H}] = e^2 \int d^4x \, {\rm tr}[A_\mu, \vec{H}]^\dagger [A_\mu, \vec{H}] \\
&= e^2 \int d^4x \sum_{\alpha,\beta} A_\mu^{\alpha *} A_\mu^\beta \vec{\alpha} \cdot \vec{\beta} {\rm tr}(E_\alpha^\dagger E_\beta) = \frac{e^2}{2} \int d^4x \sum_{\alpha=1}^{N_c(N_c-1)} |A_\mu^\alpha|^2, \qquad (15)
\end{aligned}
$$

which expresses the total amount of the off-diagonal gluon component. Here, we have used the Cartan decomposition, $A_\mu \equiv A_\mu^a T^a = \vec{A}_\mu \cdot \vec{H} + \sum_{\alpha=1}^{N_c(N_c-1)} A_\mu^\alpha E^\alpha$; $\vec{H} \equiv (T_3, T_8, \cdots, T_{N_c^2-1})$ is the Cartan subalgebra, and $E^\alpha (\alpha = 1, 2, \cdots, N_c^2 - N_c)$ denotes the raising or lowering operator. In this definition with $\hat{D}_\mu$, the gauge transformation property of $R_{\rm ch}$ becomes quite clear, because the $\mathrm{SU}(N_c)$ covariant derivative $\hat{D}_\mu$ obeys the simple adjoint gauge transformation, $\hat{D}_\mu \to \Omega \hat{D}_\mu \Omega^\dagger$, with the $\mathrm{SU}(N_c)$ gauge function $\Omega \in \mathrm{SU}(N_c)$. By the $\mathrm{SU}(N_c)$ gauge transformation, $R_{\rm ch}$ is transformed as

$$
\begin{aligned}
R_{\rm ch} \to R_{\rm ch}^\Omega &= \int d^4x \, {\rm tr} \left([\Omega \hat{D}_\mu \Omega^\dagger, \vec{H}]^\dagger [\Omega \hat{D}_\mu \Omega^\dagger, \vec{H}] \right) \\
&= \int d^4x \, {\rm tr} \left([\hat{D}_\mu, \Omega^\dagger \vec{H} \Omega]^\dagger [\hat{D}_\mu, \Omega^\dagger \vec{H} \Omega] \right), \qquad (16)
\end{aligned}
$$

and hence the residual symmetry corresponding to the invariance of $R_{\rm ch}$ is found to be $\mathrm{U}(1)_{\rm local}^{N_c-1} \times P_{\rm global}^{N_c} \subset \mathrm{SU}(N_c)_{\rm local}$, where $P_{\rm global}^{N_c}$ denotes the global Weyl group relating to the permutation of the N_c basis in the fundamental representation. In fact, one finds $\omega^\dagger \vec{H} \omega = \vec{H}$ for $\omega = e^{-i\vec{\varphi}(x)\cdot\vec{H}} \in \mathrm{U}(1)_{\rm local}^{N_c-1}$, and the global Weyl transformation by $W \in P_{\rm global}^{N_c}$ only exchanges the permutation of the nontrivial root $\vec{\alpha}_j$ and never changes $R_{\rm ch}$. In the MA gauge, arbitrary gauge transformation by $^\forall\Omega \in G$ is to increase $R_{\rm ch}$ as $R_{\rm ch}^\Omega \ge R_{\rm ch}$. Considering arbitrary infinitesimal gauge transformation $\Omega = e^{i\varepsilon} \simeq 1 + i\varepsilon$ with $^\forall \varepsilon \in su(N_c)$, one finds $\Omega^\dagger \vec{H} \Omega \simeq \vec{H} + i[\vec{H}, \varepsilon]$ and

$$
\begin{aligned}
R_{\rm ch}^\Omega &\simeq R_{\rm ch} + 2i \int d^4x {\rm tr} \left([\hat{D}_\mu, [\vec{H}, \varepsilon]][\hat{D}_\mu, \vec{H}] \right) \\
&= R_{\rm ch} + 2i \int d^4x {\rm tr} \left(\varepsilon [\vec{H}, [\hat{D}_\mu, [\hat{D}_\mu, \vec{H}]]] \right). \qquad (17)
\end{aligned}
$$

In the MA gauge, the extremum condition of $R_{\rm ch}^\Omega$ on $^\forall \varepsilon \in su(N_c)$ provides

$$
[\vec{H}, [\hat{D}_\mu, [\hat{D}_\mu, \vec{H}]]] = 0, \qquad (18)
$$

which leads to $\sum_\mu (i\partial_\mu \pm eA^3_\mu)A^\pm_\mu = 0$ for the N_c=2 case. Thus, the operator Φ to be diagonalized in the MA gauge is found to be

$$\Phi[A_\mu] = [\hat{D}_\mu, [\hat{D}_\mu, \vec{H}]] \tag{19}$$

in the continuum theory.

IV. MICROSCOPIC ABELIAN DOMINANCE IN THE MA GAUGE

In the abelian gauge, the diagonal and the off-diagonal gluons play different roles in terms of the residual abelian gauge symmetry: the diagonal gluon behaves as the abelian gauge field, while off-diagonal gluons behave as charged matter fields [40]. Under the $\mathrm{U}(1)_3$ gauge transformation by $\omega = \exp(-i\varphi\frac{\tau_3}{2}) \in \mathrm{U}(1)_3$, one finds

$$A^3_\mu \to (A^\omega_\mu)^3 = A^3_\mu + \frac{1}{e}\partial_\mu\varphi \tag{20}$$

$$A^\pm_\mu \to (A^\omega_\mu)^\pm = A^\pm_\mu e^{\pm i\varphi} \tag{21}$$

with $A^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu)$. The abelian projection is simply defined as the replacement of the gluon field $A_\mu = A^a_\mu \frac{\tau^a}{2} \in su(2)$ by the diagonal part $\mathcal{A}_\mu \equiv A^3_\mu \frac{\tau^3}{2} \in u(1)_3 \subset su(2)$.

We call "abelian dominance for an operator $\hat{O}[A_\mu]$", when the expectation value $\langle \hat{O} \rangle$ is almost unchanged by the abelian projection $A_\mu \to \mathcal{A}_\mu$ as $\langle \hat{O}[A_\mu] \rangle \simeq \langle \hat{O}[\mathcal{A}_\mu] \rangle_{\mathrm{A.G.}}$, when $\langle \ \rangle_{\mathrm{A.G.}}$ denotes the expectation value in the abelian gauge. Ordinary abelian dominance is observed for the long-distance physics in the MA gauge, and this would be physically interpreted as the effective-mass generation of the off-diagonal gluon induced by the MA gauge fixing [48,49].

In the lattice formalism, the SU(2) link-variable $U_\mu(s)$ can be factorized as

$$\begin{aligned} U_\mu(s) &= M_\mu(s) u_\mu(s) && \in G \\ M_\mu(s) &= \exp\left(i\{\tau_1\theta^1_\mu(s) + \tau_2\theta^2_\mu(s)\}\right) && \in G/H, \\ u_\mu(s) &= \exp\left(i\tau^3\theta^3_\mu(s)\right) && \in H \end{aligned} \tag{22}$$

with respect to the Cartan decomposition of $G = G/H \times H$ into G/H =SU(2)/U(1)$_3$ and H =U(1)$_3$. Here, the abelian link variable,

$$u_\mu(s) = e^{i\tau^3\theta^3_\mu(s)} = \begin{pmatrix} e^{i\theta^3_\mu(s)} & 0 \\ 0 & e^{-i\theta^3_\mu(s)} \end{pmatrix} \qquad \in \mathrm{U}(1)_3 \subset \mathrm{SU}(2), \tag{23}$$

plays the similar role as the SU(2)-link variable $U_\mu(s) \in \mathrm{SU}(2)$ in terms of the residual $\mathrm{U}(1)_3$ gauge symmetry in the abelian gauge, and $\theta^3_\mu(s) \in (-\pi, \pi]$ corresponds to the diagonal component of the gluon in the continuum limit. On the other hand, the off-diagonal factor $M_\mu(s) \in$SU(2)/U(1)$_3$ is expressed as

$$
\begin{aligned}
M_\mu(s) &= \exp\left(i\{\tau_1\theta^1_\mu(s) + \tau_2\theta^2_\mu(s)\}\right) \\
&= \begin{pmatrix} \cos\theta_\mu(s) & -\sin\theta_\mu(s)e^{-i\chi_\mu(s)} \\ \sin\theta_\mu(s)e^{i\chi_\mu(s)} & \cos\theta_\mu(s) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{1-|c_\mu(s)|^2} & -c^*_\mu(s) \\ c_\mu(s) & \sqrt{1-|c_\mu(s)|^2} \end{pmatrix}
\end{aligned} \tag{24}
$$

with $\theta_\mu(s) \equiv \mathrm{mod}_{\frac{\pi}{2}}\sqrt{(\theta^1_\mu)^2+(\theta^2_\mu)^2} \in [0, \frac{\pi}{2}]$ and $\chi_\mu(s) \in (-\pi, \pi]$. Near the continuum limit, the off-diagonal elements of $M_\mu(s)$ correspond to the off-diagonal gluon components. Under the residual $\mathrm{U}(1)_3$ gauge transformation by $\omega(s) = e^{-i\varphi(s)\frac{\tau_3}{2}} \in \mathrm{U}(1)_3$, $u_\mu(s)$ and $M_\mu(s)$ are transformed as

$$
u_\mu(s) \to u^\omega_\mu(s) = \omega(s)u_\mu(s)\omega^\dagger(s+\hat{\mu}) \qquad \in H \tag{25}
$$

$$
M_\mu(s) \to M^\omega_\mu(s) = \omega(s)M_\mu(s)\omega^\dagger(s) \qquad \in G/H \tag{26}
$$

so as to keep $M^\omega_\mu(s)$ belong G/H. Accordingly, $\theta^3_\mu(s)$ and $c_\mu(s) \in \mathbf{C}$ are transformed as

$$
\theta^3_\mu(s) \to \theta^{3\omega}_\mu(s) = \mathrm{mod}_{2\pi}[\theta^3_\mu(s) + \{\varphi(s+\hat{\mu}) - \varphi(s)\}/2] \tag{27}
$$

$$
c_\mu(s) \to c^\omega_\mu(s) = c_\mu(s)e^{i\varphi(s)}. \tag{28}
$$

Thus, on the residual $\mathrm{U}(1)_3$ gauge symmetry, $u_\mu(s)$ behaves as the $\mathrm{U}(1)_3$ lattice gauge field, and $\theta^3_\mu(s)$ behaves as the $\mathrm{U}(1)_3$ gauge field in the continuum limit. On the other hand, $M_\mu(s)$ and $c_\mu(s)$ behave as the charged matter field in terms of the residual $\mathrm{U}(1)_3$ gauge symmetry, which is similar to the charged weak boson $W^\pm_\mu$ in the standard model.

In this parameterization (22), there are two U(1)-structures embedded in SU(2) corresponding to $e^{i\theta^3_\mu}$ and $e^{i\tilde{\chi}_\mu}$. To clarify this structure, we reparametrize the SU(2) link variable as

$$
U_\mu(s) = \begin{pmatrix} \cos\theta_\mu e^{i\theta^3_\mu} & -\sin\theta_\mu e^{-i\tilde{\chi}_\mu} \\ \sin\theta_\mu e^{i\tilde{\chi}_\mu} & \cos\theta_\mu e^{-i\theta^3_\mu} \end{pmatrix}, \tag{29}
$$

or equivalently

$$
\begin{aligned}
U^0_\mu &= \cos\theta_\mu\cos\theta^3_\mu, \qquad & U^1_\mu &= \sin\theta_\mu\sin\tilde{\chi}_\mu, \\
U^3_\mu &= \cos\theta_\mu\sin\theta^3_\mu, \qquad & U^2_\mu &= \sin\theta_\mu\cos\tilde{\chi}_\mu,
\end{aligned} \tag{30}
$$

with $\tilde{\chi}_\mu \equiv \chi_\mu + \theta^3_\mu$. The range of the angle variable can be redefined as $0 \le \theta_\mu \le \frac{\pi}{2}$ and $-\pi < \theta^3_\mu, \tilde{\chi}_\mu \le \pi$. Here, $(U^0_\mu, U^1_\mu, U^2_\mu, U^3_\mu)$ forms an element of the 3-dimensional hypersphere $S^3 \simeq$SU(2), because of $(U^0_\mu)^2 + (U^1_\mu)^2 + (U^2_\mu)^2 + (U^3_\mu)^2 = 1$. For a fixed θ_μ, both (U^0_μ, U^3_μ) and (U^1_μ, U^2_μ) form the two $S^1 \simeq$U(1) subgroups embedded in S^3 in a symmetric manner. From the parametrization in Eq.(29), the SU(2) measure can be easily found as

$$\int dU_\mu \equiv \int dU_\mu^0 U_\mu^1 U_\mu^2 U_\mu^3 \;\; \delta\,(\sum_{a=0}^{3}(\,U_\mu^a)^2 - 1)$$
$$= \frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}} d\theta_\mu \sin\theta_\mu \cos\theta_\mu \int_{-\pi}^{\pi} d\tilde{\chi}_\mu \int_{-\pi}^{\pi} d\theta_\mu^3. \tag{31}$$

In the lattice formalism, the abelian projection is defined by replacing the SU(2) link variable $U_\mu(s) \in$ SU(2) by the abelian link variable $u_\mu(s) \in$ U(1)$_3$.

In the MA gauge, the off-diagonal gluon component is strongly suppressed, and the SU(2) link variable is expected to be U(1)$_3$-like as $U_\mu(s) \simeq u_\mu(s)$ in the relevant gauge configuration. In the quantitative argument, this can be expected as $\langle U_\mu(s) u_\mu^\dagger(s)\rangle_{\rm MA} \simeq 1$, where $\langle\ \rangle_{\rm MA}$ denotes the expectation value in the MA gauge. In order to estimate the difference between $U_\mu(s)$ and $u_\mu(s)$, we introduce the "abelian projection rate" $R_{\rm Abel}$ [50,51], which is defined as the overlapping factor as

$$R_{\rm Abel}(s,\mu) \equiv \frac{1}{2}{\rm Re\ tr}\{U_\mu(s) u_\mu^\dagger(s)\} = \frac{1}{2}{\rm Re\ tr}M_\mu(s) = \cos\theta_\mu(s). \tag{32}$$

This definition of $R_{\rm Abel}$ is inspired by the ordinary "distance" between two matrices $A, B \in {\rm GL}(N, \mathbf{C})$ defined as $d^2(A,B) \equiv \frac{1}{2}{\rm tr}\{(A-B)^\dagger(A-B)\}$ [52], which leads to $d^2(A,B) = 2 - {\rm Re\ tr}(AB^\dagger)$ for $A, B \in$SU(2). The similarity between $U_\mu(s)$ and $u_\mu(s)$ can be quantitatively measured in terms of the "distance" between them. For instance, if $\cos\theta_\mu(s) = 1$, the SU(2) link variable becomes completely abelian as

$$U_\mu(s) = \begin{pmatrix} e^{i\theta_\mu^3} & 0 \\ 0 & e^{-i\theta_\mu^3} \end{pmatrix},$$

while, if $\cos\theta_\mu(s) = 0$, it becomes completely off-diagonal as

$$U_\mu(s) = \begin{pmatrix} 0 & -e^{-i\tilde{\chi}_\mu} \\ e^{i\tilde{\chi}_\mu} & 0 \end{pmatrix}.$$

We show in Fig.2(a) the probability distribution $P(R_{\rm Abel})$ of the abelian projection rate $R_{\rm Abel}(s,\mu) \equiv \cos\theta_\mu(s)$ in the MA gauge. Here, $\langle R_{\rm Abel}\rangle_{\beta=0}$ in the strong coupling limit ($\beta = 0$) [50,51] is analytically calculable as

$$\langle R_{\rm Abel}\rangle_{\beta=0} = \langle \cos\theta_\mu(s)\rangle_{\beta=0} = \frac{\int dU_\mu(s)\cos\theta_\mu(s)}{\int dU_\mu(s)}$$
$$= \frac{\int_0^{\frac{\pi}{2}} d\theta_\mu(s)\sin\theta_\mu(s)\cos^2\theta_\mu(s)}{\int_0^{\frac{\pi}{2}} d\theta_\mu(s)\sin\theta_\mu(s)\cos\theta_\mu(s)} = \frac{2}{3}, \tag{33}$$

using Eq.(31). In the MA gauge, $R_{\rm Abel}$ approaches to unity as shown in Fig.2(a). The off-diagonal component of the SU(2) link variable is forced to be reduced. As a typical example, one obtains $\langle R_{\rm Abel}\rangle_{\rm MA} \simeq 0.926$ on 16^4 lattice with $\beta = 2.4$. We show also

the abelian projection rate $\langle R_{\rm Abel}\rangle_{\rm MA}$ as the function of β in Fig.2(b). For larger β, $\langle\cos\theta_\mu(s)\rangle_{\rm MA}$ becomes slightly larger. Without gauge fixing, the average $\langle R_{\rm Abel}\rangle$ is found to be about $\frac{2}{3}$ without dependence on β. In the continuum limit in the MA gauge, $U_\mu^1(s)$ and $U_\mu^2(s)$ become at most $O(a)$, and therefore $\langle R_{\rm Abel}\rangle_{\rm MA}$ approaches to unity as $\langle R_{\rm Abel}\rangle_{\rm MA} = 1 + O(a^2)$ due to the trivial dominance of $U_\mu^0(s)$, which differs from abelian dominance in the physical sense. The remarkable feature of the MA gauge is the high abelian projection rate as $\langle R_{\rm Abel}\rangle_{\rm MA} \simeq 1$ in the whole region of β. In fact, we find $\langle R_{\rm Abel}\rangle_{\rm MA} \simeq 0.88$ even for the strong coupling limit $\beta = 0$, where the original link variable U_μ is completely random. Thus, abelian dominance for the link variable U_μ is observed at any scale in the MA gauge.

To understand the origin of the high abelian projection rate as $\langle R_{\rm Abel}\rangle_{\rm MA} \simeq 1$, we estimate the lower bound of $\langle R_{\rm Abel}\rangle_{\rm MA}$ in the MA gauge using the statistical consideration. The MA gauge maximizes

$$R_{\rm MA}[U_\mu] \equiv \sum_{s,\mu} {\rm tr}\{U_\mu(s)\tau_3 U_\mu^\dagger(s)\tau_3\} = {\rm tr}(\tau_3 \sum_{s,\mu} \hat\phi_\mu(s)), \tag{34}$$

where $\hat\phi_\mu(s) \equiv U_\mu(s)\tau_3 U_\mu^\dagger(s)$ is an $su(2)$ element satisfying $\hat\phi_\mu^2 = 1$. Denoting $\hat\phi_\mu(s) = \hat\phi_\mu^a(s)\tau^a$, we parameterize the 3-dimensional unit vectors $\vec\phi_\mu \equiv (\hat\phi_\mu^1, \hat\phi_\mu^2, \hat\phi_\mu^3) \in S^2$ $(\mu = 1,2,3,4)$ as $\vec\phi_\mu = (\sin 2\theta_\mu \cos\chi_\mu, \sin 2\theta_\mu \sin\chi_\mu, \cos 2\theta_\mu)$ using Eqs.(22) and (24). The MA gauge maximizes the third component $\hat\phi_\mu^3$ using the gauge transformation. Under the local gauge transformation by $V(s) \equiv 1 + \{V(s_0) - 1\}\delta_{ss_0} \in {\rm SU}(2)$, $\hat\phi_\mu(s_0)$ is transformed as the unitary transformation,

$$\hat\phi_\mu(s_0) \to \hat\phi'_\mu(s_0) \equiv V(s_0)\hat\phi_\mu(s_0)V^{-1}(s_0), \tag{35}$$

which leads to a simple rotation of the unit vectors $\vec\phi_\mu$. In the MA gauge, $\sum_{s,\mu}\vec\phi_\mu$ is "polarized" along the positive third direction. On the 4-dimension lattice with N sites, $4N$ unit vectors $\vec\phi_\mu(s)$ are maximally polarized by N gauge functions $V(s)$ in the MA gauge. Then, $\langle R_{\rm Abel}\rangle_{\rm MA}$ is expressed as the maximal "polarization rate" of $4N$ unit vectors $\vec\phi_\mu$ by suitable N gauge functions $V(s)$. On the average, this estimation of $\langle R_{\rm Abel}\rangle_{\rm MA}$ is approximately given by the estimation of the maximal polarization rate of 4 unit vectors $\vec\phi_\mu$ by a suitable rotation with $V \in {\rm SU}(2)$. The lower bound of $\langle R_{\rm Abel}\rangle_{\rm MA}$ is obtained from the strong-coupling system with $\beta = 0$, where link variables $U_\mu(s)$ are completely random. Accordingly, $\vec\phi_\mu$ can be regarded as random unit vectors on S^2. The maximal "polarization" of 4 unit vectors $\vec\phi_\mu$ is realized by the rotation which moves $\vec\phi \equiv \sum_{\mu=1}^4 \vec\phi_\mu / |\sum_{\mu=1}^4 \vec\phi_\mu|$ to the unit vector $\vec\phi^R \equiv (0,0,1)$ in third direction. Here, $\cos 2\theta_\mu^R$ after the rotation is identical to the inner product between $\vec\phi_\mu$ and $\vec\phi$, because of $\vec\phi\cdot\vec\phi_\mu = \vec\phi^R\cdot\vec\phi_\mu^R = (\hat\phi_\mu^R)^3 = \cos 2\theta_\mu^R$. Then, we estimate $\langle R_{\rm Abel}\rangle_{\rm MA} = \langle\cos\theta_\mu\rangle_{\rm MA}$ at $\beta = 0$ as

$$
\begin{aligned}
\langle\cos\theta_\mu\rangle_{\rm MA}^{\beta=0} &\simeq \{\prod_{\mu=1}^{4}\int dU_\mu\}\left(\frac{1}{4}\sum_{\mu=1}^{4}\cos\theta_\mu^R\right) \\
&= \{\prod_{\mu=1}^{4}\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta_\mu\cos\theta_\mu\sin\theta_\mu\int_{-\pi}^{\pi} d\chi_\mu\}\left(\frac{1}{4}\sum_{\mu=1}^{4}\cos\{\frac{1}{2}\cos^{-1}(\vec{\phi}\cdot\vec{\phi}_\mu)\}\right). \qquad (36)
\end{aligned}
$$

Using this estimation (36), we obtain $\langle R_{\rm Abel}\rangle_{\rm MA}\simeq 0.844$, which is close to the lattice result $\langle R_{\rm Abel}\rangle\simeq 0.88$ in the strong coupling limit ($\beta=0$). Such a high abelian rate $\langle R_{\rm Abel}\rangle_{\rm MA}$ in the MA gauge would provide a microscopic basis of abelian dominance for the infrared physics.

V. SEMI-ANALYTICAL PROOF OF ABELIAN DOMINANCE FOR CONFINEMENT FORCE

Abelian dominance and monopole dominance for the nonperturbative phenomena are numerically observed in the MA gauge in the lattice QCD simulations [35,39,44–49]. As for confinement, monopole dominance, which means the dominant role of the magnetic current k_μ than that of the electric current j_μ, seems trivial if abelian dominance holds, because j_μ does not provide the electric confinement in 1+3 dimension. Then, as for confinement, abelian dominance, which means the dominant role of the diagonal element than that of the off-diagonal element, is the nontrivial interesting phenomenon observed in the MA gauge in the lattice QCD. In this section, we study the origin of abelian dominance on the string tension as the confinement force in a semi-analytical manner, considering the relation with *microscopic abelian dominance* on the link variable [51].

In the MA gauge, the diagonal element $\cos\theta_\mu(s)$ in $M_\mu(s)$ is maximized by the gauge transformation as large as possible. For instance, the abelian projection rate is almost unity as $R_{\rm Abel}=\langle\cos\theta_\mu(s)\rangle_{\rm MA}\simeq 0.93$ at $\beta=2.4$. Then, the *off-diagonal element* $e^{i\chi_\mu(s)}\sin\theta_\mu(s)$ *is forced to take a small value in the MA gauge* due to the factor $\sin\theta_\mu(s)$, and therefore the approximate treatment on the off-diagonal element would be allowed in the MA gauge. Moreover, the *angle variable* $\chi_\mu(s)$ *is not constrained* by the MA gauge-fixing condition at all, and *tends to take a random value* besides the residual $U(1)_3$ gauge degrees of freedom. Hence, $\chi_\mu(s)$ can be regarded as a *random angle variable* on the treatment of $M_\mu(s)$ in the MA gauge in a good approximation.

Let us consider the Wilson loop $\langle W_C[U_\mu(s)]\rangle\equiv\langle{\rm tr}\prod_C U_\mu(s)\rangle=\langle{\rm tr}\prod_C\{M_\mu(s)u_\mu(s)\}\rangle$ in the MA gauge. In calculating $\langle W_C[U_\mu(s)]\rangle$, the expectation value of $e^{i\chi_\mu(s)}$ in $M_\mu(s)$ vanishes as

$$
\langle e^{i\chi_\mu(s)}\rangle\simeq\int_0^{2\pi}d\chi_\mu(s)\exp\{i\chi_\mu(s)\}=0, \qquad (37)
$$

when $\chi_\mu(s)$ behaves as a *random angle variable.* Then, within the random-variable approximation for $\chi_\mu(s)$, the *off-diagonal factor* $M_\mu(s)$ appearing in $\langle W_C[U_\mu(s)]\rangle$ is simply reduced as a c-number factor, $M_\mu(s) \to \cos\theta_\mu(s)\ \mathbf{1}$, and therefore the SU(2) link variable $U_\mu(s)$ in the Wilson loop $\langle W_C[U_\mu(s)]\rangle$ is simplified as a *diagonal matrix,*

$$U_\mu(s) \equiv M_\mu(s)u_\mu(s) \to \cos\theta_\mu(s)u_\mu(s). \tag{38}$$

Then, for the $I \times J$ rectangular C, the Wilson loop $W_C[U_\mu(s)]$ in the MA gauge is approximated as

$$\begin{aligned}\langle W_C[U_\mu(s)]\rangle &\equiv \langle \mathrm{tr} \prod_{i=1}^{L} U_{\mu_i}(s_i)\rangle \simeq \langle \prod_{i=1}^{L} \cos\theta_{\mu_i}(s_i) \cdot \mathrm{tr} \prod_{j=1}^{L} u_{\mu_j}(s_j)\rangle_{\mathrm{MA}} \\ &\simeq \langle \exp\{\sum_{i=1}^{L} \ln(\cos\theta_{\mu_i}(s_i))\}\rangle_{\mathrm{MA}}\ \langle W_C[u_\mu(s)]\rangle_{\mathrm{MA}} \\ &\simeq \exp\{L\langle \ln(\cos\theta_\mu(s))\rangle_{\mathrm{MA}}\}\ \langle W_C[u_\mu(s)]\rangle_{\mathrm{MA}},\end{aligned} \tag{39}$$

where $L \equiv 2(I+J)$ denotes the perimeter length and $W_C[u_\mu(s)] \equiv \mathrm{tr} \prod_{i=1}^{L} u_{\mu_i}(s_i)$ the abelian Wilson loop. Here, we have replaced $\sum_{i=1}^{L} \ln\{\cos(\theta_{\mu_i}(s_i)\}$ by its average $L\langle \ln\{\cos\theta_\mu(s)\}\rangle_{\mathrm{MA}}$ *in a statistical sense*, and such a statistical treatment becomes more accurate for larger I, J and becomes exact for infinite I, J.

In this way, we derive a simple estimation as

$$W_C^{\mathrm{off}} \equiv \langle W_C[U_\mu(s)]\rangle / \langle W_C[u_\mu(s)]\rangle_{\mathrm{MA}} \simeq \exp\{L\langle \ln(\cos\theta_\mu(s))\rangle_{\mathrm{MA}}\} \tag{40}$$

for the *contribution of the off-diagonal gluon element to the Wilson loop.* From this analysis, the contribution of off-diagonal gluons to the Wilson loop is expected to obey the *perimeter law* in the MA gauge for large loops, where the statistical treatment would be accurate.

Now, we study the behavior of the off-diagonal contribution $W_C^{\mathrm{off}} \equiv \langle W_C[U_\mu(s)]\rangle / \langle W_C[u_\mu(s)]\rangle_{\mathrm{MA}}$ in the MA gauge using the lattice QCD, considering the theoretical estimation Eq.(40). As shown in Fig.3, we find that W_C^{off} seems to obey the *perimeter law* for the Wilson loop with $I, J \geq 2$ in the MA gauge in the lattice QCD simulation with $\beta = 2.4$ and 16^4. We find also that the behavior on W_C^{off} as the function of L is well reproduced by the above analytical estimation with the *microscopic information* on the diagonal factor $\cos\theta_\mu(s)$ as $\langle \ln\{\cos\theta_\mu(s)\}\rangle_{\mathrm{MA}} \simeq -0.082$ for $\beta = 2.4$. Thus, the off-diagonal contribution W_C^{off} to the Wilson loop obeys the perimeter law in the MA gauge, and therefore the *abelian Wilson loop* $\langle W_C[u_\mu(s)]\rangle_{\mathrm{MA}}$ should obey the *area law* as well as the SU(2) Wilson loop $W_C[U_\mu(s)]$. From Eq.(40), the off-diagonal contribution to the string tension vanishes as

$$\begin{aligned}\Delta\sigma &\equiv \sigma_{\mathrm{SU}(2)} - \sigma_{\mathrm{Abel}} \\ &\equiv -\lim_{R,T\to\infty}\frac{1}{RT}\ln\langle W_{R\times T}[U_\mu(s)]\rangle + \lim_{R,T\to\infty}\frac{1}{RT}\ln\langle W_{R\times T}[u_\mu(s)]\rangle_{\mathrm{MA}} \\ &\simeq -2\langle \ln\{\cos\theta_\mu(s)\}\rangle_{\mathrm{MA}}\lim_{R,T\to\infty}\frac{R+T}{RT} = 0.\end{aligned} \tag{41}$$

Thus, *abelian dominance for the string tension*, $\sigma_{\mathrm{SU}(2)} = \sigma_{\mathrm{Abel}}$, can be proved in the MA gauge by replacing the off-diagonal angle variable $\chi_\mu(s)$ as a random variable.

The analytical relation in Eq.(40) indicates also that the finite size effect on R and T in the Wilson loop leads to the deviation between the SU(2) string tension $\sigma_{\mathrm{SU}(2)}$ and the abelian string tension σ_{Abel} as $\sigma_{\mathrm{SU}(2)} > \sigma_{\mathrm{Abel}}$ in the actual lattice QCD simulations. Here, we consider this deviation $\Delta\sigma \equiv \sigma_{\mathrm{SU}(2)} - \sigma_{\mathrm{Abel}}$ in some detail. Similar to the SU(2) inter-quark potential $V_{\mathrm{SU}(2)}(r)$ from $\langle W_{\mathrm{SU}(2)}\rangle \equiv \langle W[U_\mu(s)]\rangle$, we define the abelian inter-quark potential $V_{\mathrm{Abel}}(r)$ and the off-diagonal contribution $V_{\mathrm{off}}(r)$ of the potential from $\langle W_{\mathrm{Abel}}\rangle \equiv \langle W[u_\mu(s)]\rangle$ and W_{off}, respectively,

$$\begin{aligned}V_{\mathrm{SU}(2)}(r) &\equiv -\frac{1}{Ta}\ln\langle W_{\mathrm{SU}(2)}(R\times T)\rangle, \\ V_{\mathrm{Abel}}(r) &\equiv -\frac{1}{Ta}\ln\langle W_{\mathrm{Abel}}(R\times T)\rangle, \\ V_{\mathrm{off}}(r) &\equiv -\frac{1}{Ta}\ln W_{\mathrm{off}}(R\times T) = -\frac{1}{Ta}\ln\frac{\langle W_{\mathrm{SU}(2)}(R\times T)\rangle}{\langle W_{\mathrm{Abel}}(R\times T)\rangle} \\ &= V_{\mathrm{SU}(2)}(r) - V_{\mathrm{Abel}}(r),\end{aligned} \tag{42}$$

wehre $r \equiv Ra$ denotes the inter-quark distance in the physical unit. We show in Fig.4 $V_{\mathrm{SU}(2)}(r)$, $V_{\mathrm{Abel}}(r)$ and $V_{\mathrm{off}}(r)$ extracted from the Wilson loop with $T = 7$ in the lattice QCD simulation with $\beta = 2.4$ and 16^4. As shown in Fig.4, the lattice result for $V_{\mathrm{off}}(r)$ seems to be reproduced by the theoretical estimation obtained from Eq.(40),

$$V_{\mathrm{off}}(r) = V_{\mathrm{SU}(2)}(r) - V_{\mathrm{Abel}}(r) \simeq -\frac{2(R+T)}{Ta}\langle \ln(\cos\theta_\mu(s))\rangle_{\mathrm{MA}} \tag{43}$$

using the microscopic information of $\langle \ln(\cos\theta_\mu(s))\rangle_{\mathrm{MA}} = -0.082$ at $\beta = 2.4$. From the slope of $V_{\mathrm{off}}(r)$ in Eq.(43), we can estimate $\Delta\sigma \equiv \sigma_{\mathrm{SU}(2)} - \sigma_{\mathrm{Abel}}$ in the physical unit as

$$\begin{aligned}\Delta\sigma \equiv \sigma_{\mathrm{SU}(2)} - \sigma_{\mathrm{Abel}} &\simeq -2\langle \ln(\cos\theta_\mu(s))\rangle_{\mathrm{MA}}\frac{1}{Ta^2} \\ &= -\langle \ln(1-\sin^2\theta_\mu(s))\rangle_{\mathrm{MA}}\frac{1}{Ta^2}.\end{aligned} \tag{44}$$

In the MA gauge, $\sin^2\theta_\mu(s)$ takes a small value and can be treated in a perturbation manner so that one finds

$$\Delta\sigma \simeq \langle \sin^2\theta_\mu(s)\rangle_{\mathrm{MA}}\frac{1}{Ta^2} = \langle (U_\mu^1(s))^2 + (U_{\mu 2}^2(s))^2\rangle_{\mathrm{MA}}\frac{1}{Ta^2}. \tag{45}$$

Near the continuum limit $a \simeq 0$, we find $U_\mu^a \simeq aeA_\mu^a/2$ (a=1,2,3) from $U_\mu = e^{iaeA_\mu^a \tau^a/2}$, and then we derive the relation between $\Delta\sigma$ and the off-diagonal gluon in the MA gauge as

$$\Delta\sigma \simeq \frac{1}{4T}\langle (eA_\mu^1)^2 + (eA_\mu^2)^2 \rangle_{\rm MA} = \frac{a}{4t}\langle (eA_\mu^1)^2 + (eA_\mu^2)^2 \rangle_{\rm MA}, \tag{46}$$

where $t \equiv Ta$ is the temporal length of the Wilson loop in the physical unit. In Eq.(46), $\langle (eA_\mu^1)^2 + (eA_\mu^2)^2 \rangle_{\rm MA}$ is the off-diagonal gluon-field fluctuation, and is strongly suppressed in the MA gauge by its definition. It would be interesting to note that microscopic abelian dominance or the suppression of off-diagonal gluons in the MA gauge is directly connected to reduction of the deviation $\Delta\sigma$ in Eq.(46). Since $\langle (eA_\mu^1)^2 + (eA_\mu^2)^2 \rangle_{\rm MA}$ is a local continuum quantity, it is to be independent on both a and t. Hence, the deviation $\Delta\sigma$ between the SU(2) string tension $\sigma_{\rm SU(2)}$ and the abelian string tension $\sigma_{\rm Abel}$ can be removed by taking the large Wilson loop as $t \to \infty$ or the small mesh as $a \to 0$ with fixed t.

Finally in this section, we study the origin of abelian dominance in the MA gauge in terms of the properties of the off-diagonal element

$$c_\mu(s) \equiv e^{i\chi_\mu(s)} \sin\theta_\mu(s) \tag{47}$$

of $M_\mu(s)$ in the link variable $U_\mu(s)$, considering the validity of the random-variable approximation for $\chi_\mu(s)$ in the MA gauge. *In the above treatment, the contribution of the off-diagonal element in the link variable $U_\mu(s)$ is completely dropped off, and its effect indirectly remains as the appearance of the c-number factor $\cos\theta_\mu(s)$ in the link variable. Such a reduction of the contribution of the off-diagonal elements is brought by the two relevant features on the two local variables, $\theta_\mu(s)$ and $\chi_\mu(s)$, in the MA gauge. One is the microscopic abelian dominance as $\langle \cos\theta_\mu(s) \rangle_{\rm MA} \simeq 1$ in the MA gauge, and the other is the randomness of the off-diagonal variable $\chi_\mu(s)$.*

1. In the MA gauge, the microscopic abelian dominance holds as $\langle \cos\theta_\mu(s) \rangle_{\rm MA} \simeq 1$, and the absolute value of the off-diagonal element $|c_\mu(s)| = |\sin\theta_\mu(s)|$ is strongly reduced. Such a tendency becomes more significant as β increases.

2. The off-diagonal angle variable $\chi_\mu(s)$ is not constrained by the MA gauge-fixing condition at all, and tends to be a random variable. In fact, $\chi_\mu(s)$ is affected only by the action factor $e^{-\beta S_{\rm QCD}}$ in the QCD generating functional, but the effect of the action to $\chi_\mu(s)$ is quite weaken due to the small factor $\sin\theta_\mu(s)$ in the MA gauge. The randomness of $\chi_\mu(s)$ tends to vanish the contribution of the off-diagonal elements.

Here, the randomness of the off-diagonal angle-variable $\chi_\mu(s)$ is closely related to the microscopic abelian dominance. In fact, the randomness of $\chi_\mu(s)$ is controlled only by the action factor $e^{-\beta S_{\rm QCD}}$ in the QCD generating functional, however the effect of the action to $\chi_\mu(s)$ is quite weaken due to the small factor $\sin\theta_\mu(s)$ in the MA gauge, because $\chi_\mu(s)$ always accompanies $\sin\theta_\mu(s)$ in the link variable $U_\mu(s)$. Near the strong-coupling limit $\beta \simeq 0$, the action factor $e^{-\beta S_{\rm QCD}}$ brings almost no constraint on $\chi_\mu(s)$ in the MA gauge. The independence of $\chi_\mu(s)$ from the action factor is enhanced by the small factor $\sin\theta_\mu(s)$ accompanying $\chi_\mu(s)$. Hence, $\chi_\mu(s)$ behaves as a random angle-variable almost exactly, and the contribution of the off-diagonal element is expected to disappear in the strong-coupling region. As β increases, the action factor $e^{-\beta S_{\rm QCD}}$ becomes relevant and will reduce the randomness of $\chi_\mu(s)$ to some extent. Near the continuum limit $\beta \to \infty$, however, the factor $\sin\theta_\mu(s)$ tends to approach 0 in the MA gauge as shown in Fig.2(b), and hence such a constraint on $\chi_\mu(s)$ from the action is largely reduced, and the strong randomness of $\chi_\mu(s)$ is expected to hold there. Moreover, the reduction of the absolute value $|c_\mu(s)| = |\sin\theta_\mu(s)|$ itself further reduces the contribution of the off-diagonal element $|c_\mu(s)|$ in the MA gauge.

Now, we examine the randomness of $\chi_\mu(s)$ using the lattice QCD simulation. It should be noted that the residual $U(1)_3$ gauge degrees of freedom should be fixed to extract $\chi_\mu(s)$ itself, because $\chi_\mu(s)$ is the $U(1)_3$ gauge variant. To this end, we add the $U(1)_3$ lattice Landau gauge [48], which maximizes

$$R = \sum_{s,\mu} Re\, {\rm tr}\, u_\mu(s) \tag{48}$$

using the residual $U(1)_3$ gauge transformation in the MA gauge. In the $U(1)_3$ Landau gauge, there remains no local symmetry and the lattice variable becomes mostly continuous and approaches to the continuum field under the constraint of the MA gauge fixing. For the test of the randomness of $\chi_\mu(s)$, we calculate the probability distributions of $\chi_\mu(s)$ and the correlation between $\chi_\mu(s)$ and $\chi_\mu(s+\hat\nu)$ in the MA gauge with the $U(1)_3$-Landau gauge. If $\chi_\mu(s)$ is a random angle variable, there is no bias on the distribution of $\chi_\mu(s)$ and no correlation is observed between $\chi_\mu(s)$ and $\chi_\mu(s+\hat\nu)$. We show in Fig.5(a) the probability distributions $P(\chi_\mu)$ and $P(\theta^3_\mu)$ at $\beta = 2.4$. Unlike $P(\theta^3_\mu)$, $P(\chi_\mu)$ is flat distribution without any structure in the whole region of β, which is necessary condition of the random angle variable. We show in Fig.5(b) the probability distribution $P(\Delta\chi)$ of the correlation

$$\Delta\chi(s) \equiv d(\chi_\mu(s), \chi_\mu(s+\hat\nu)) \equiv {\rm mod}_\pi|\chi_\mu(s) - \chi_\mu(s+\hat\nu)| \in [0,\pi], \tag{49}$$

which is the difference between two neighboring angle variables, at β=0, 1.0, 2.4, 3.0. In the strong-coupling limit $\beta = 0$, $\chi_\mu(s)$ is a completely random variable, and there

is no correlation between neighboring χ_μ. In the strong-coupling region as $\beta \le 1.0$, almost no correlation is observed between neighboring χ_μ, which suggests the strong randomness of $\chi_\mu(s)$. As a remarkable feature, the correlation between neighboring χ_μ seems weak even in the weak-coupling region as $\beta \ge 2.4$, where the action factor $e^{-\beta S_{\rm QCD}}$ becomes dominant and remaining variables $\theta_\mu^3(s)$ and $\theta_\mu(s)$ behave as continuous variables with small difference between their neighbors as $\Delta\theta_\mu^3 \simeq 0$ and $\Delta\theta_\mu \simeq 0$. Such a weak correlation of neighboring χ_μ would be originated from the reduction of the accompanying factor $\sin\theta_\mu(s)$ in the MA gauge. Moreover, in the weak-coupling region, the smallness of $\sin\theta_\mu(s)$ makes $c_\mu(s)$ more irrelevant in the MA gauge, which permits some approximation on $\chi_\mu(s)$. Thus, the random-variable approximation for $\chi_\mu(s)$ would provide a good approximation in the whole region of β in the MA gauge. To conclude, the origin of abelian dominance for confinement in the MA gauge is stemming from the strong randomness of the off-diagonal angle variable $\chi_\mu(s)$ and the strong reduction of the off-diagonal amplitude $|\sin\theta_\mu(s)|$ as the result of the MA gauge fixing.

VI. SUMMARY AND CONCLUDING REMARKS

In the 't Hooft abelian gauge, QCD is reduced into an abelian gauge theory, and the color-magnetic monopole appears as the topological object in the constrained nonabelian gauge manifold corresponding to the nontrivial homotopy group $\Pi_2(SU(N_c)/U(1)^{N_c-1}) = Z_\infty^{N_c-1}$. Hence, if off-diagonal gluons can be neglected and the monopole is condensed, the QCD vacuum in the abelian gauge is described as the abelian dual superconductor and the confinement mechanism is understood as the dual Meissner effect.

In relation with the dual Higgs picture for the confinement mechanism in QCD, we have studied the mathematical features of the abelian gauge fixing, the local gluon properties in the maximally abelian (MA) gauge, and the origin of abelian dominance for confinement in a semi-analytical manner with the help of the lattice QCD.

First, we have studied the residual symmetry in the abelian gauge, with paying attention to the global Weyl symmetry, which can remain as the relic of SU(N_c). The global Weyl symmetry provides the ambiguity on the electric and magnetic charges, and persistently remains in the MA gauge. Considering the abelian gauge fixing in terms of the coset space of the fixed gauge symmetry, we have derived the criterion on the SU(N_c)-gauge invariance of the operator in the abelian gauge: if the operator defined in the abelian gauge is invariant under the residual gauge transformation, it is also SU(N_c)-gauge invariant.

Second, in the continuum SU(N_c) QCD, we have expressed the MA gauge fixing using the SU(N_c)-covariant derivative operator. The local MA-gauge fixing condition and the composite Higgs field $\Phi[A_\mu(s)]$ to be diagonalized are naturally derived from this expression of the MA gauge.

Third, we have examined the abelian projection rate $R_{\rm Abel}$, the overlapping factor between SU(2) and abelian link variables, in the lattice formalism. In the MA gauge, we have found the high abelian projection rate as $\langle R_{\rm Abel}\rangle_{\rm MA} \simeq 1$ for the whole region of β, which means microscopic abelian dominance on the link variable. Using the statistical consideration, we have analytically estimated the lower bound of the abelian projection rate in the MA gauge as $\langle R_{\rm Abel}\rangle_{\rm MA} \geq 0.84$, which seems consistent with the lattice result $\langle R_{\rm Abel}\rangle_{\rm MA} \geq 0.88$.

Finally, we have studied abelian dominance in terms of off-diagonal gluons in the Wilson loop in the MA gauge. In the SU(2) link variable, the off-diagonal angle variable is not constrained by the MA-gauge fixing condition at all, and tends to take random values besides the residual gauge degrees of freedom. By approximating the off-diagonal angle variable as a random variable, we have proved that the contribution of off-diagonal gluons to the Wilson loop, $W_{\rm off} \equiv \langle W_{\rm SU(2)}\rangle/\langle W_{\rm Abel}\rangle_{\rm MA}$, obeys the perimeter law in the MA gauge, which is numerically confirmed using the lattice QCD Monte Carlo simulation. Thus, we have showed exact abelian dominance for the string tension as $\sigma_{\rm SU(2)} = \sigma_{\rm Abel}$, although the finite size effect of the Wilson loop in the actual lattice QCD simulation leads to small deviation as $\sigma_{\rm SU(2)} > \sigma_{\rm Abel}$.

In conclusion, we have found that the origin of abelian dominance for confinement in the MA gauge is stemming from the strong randomness of the off-diagonal angle variable $\chi_\mu(s)$ and the strong reduction of the off-diagonal amplitude $|\sin\theta_\mu(s)|$, and these two remarkable features on the local variables $\chi_\mu(s)$ and $\theta_\mu(s)$ are peculiar to the MA gauge fixing.

One of authors (H.I.) is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. One of authors (H.S.) is supported in part by Grant for Scientific Research (No.09640359) from the Ministry of Education, Science and Culture, Japan. The lattice QCD simulations have been performed on VPP500 at RIKEN and sx4 at RCNP.

REFERENCES

[1] For instance, T. P. Cheng and L. F. Li, “Gauge Theory of Elementary Particle Physics”, (Clarendon press, Oxford, 1984) 1.

[2] K. Huang, “Quarks, Leptons and Gauge Fields”, (World Scientific, Singapore, 1982) 1.

[3] Y. Nambu and G. Jona-Lasinio, Phys. Rev.**122** (1961) 345; **124** (1961) 246.

[4] K. Higashijima, Prog. Theor. Phys. Suppl. **104** (1991) 1.

[5] V. A. Miransky, “Dynamical Symmetry Breaking in Quantum Field Theories” (World Scientific, 1993) 1.

[6] A. A. Belavin, A. M. Polyakov, A. S. Shvarts, Yu. S. Tyupkin, Phys. Lett. **59B** (1975) 85.

[7] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. **B272** (1986) 457.

[8] E. V. Shuryak, Phys. Rep. **115** (1984) 151.

[9] A. W. Thomas, Adv. Nucl. Phys. **13** (1984) 1.

[10] A. Hosaka and H. Toki, Phys. Rept. **277** (1996) 65.

[11] T. Hatsuda and T. Kunihiro, Phys Rept. **247** (1994) 221.

[12] M. Bando, T. Kugo and K. Yamawaki, Phys. Rept. **164** (1988) 217.

[13] Y. Nambu, Phys. Rev. **D10** (1974) 4262.

[14] G. ’t Hooft, “High Energy Physics”, ed. A. Zichichi (Editorice Compositori, Bologna, 1975).

[15] S. Mandelstam, Phys. Rep. **C23** (1976) 245.

[16] A. A. Abrikosov, “Fundamentals of the Theory of Metals”, (North-Holland, 1988) 1.

[17] K. Sailer, T. Schoenfeld, Z. Schram, A. Schaefer and W. Greiner, Int. J. Mod. Phys. **A6** (1991) 4395.

[18] H. J. Rothe, ”Lattice Gauge Theories”, (World Scientific, 1992) 1.

[19] M. Creutz, ”Quarks, Gluons and Lattices” (Cambridge, 1983) 1.

[20] Y. Peng and R. W. Haymaker, Nucl. Phys. **B** (Proc. Suppl.) **34** (1996) 266.
R. W. Haymaker, V. Singh, Y. Peng and J. Wosiek, Phys. Rev.**D53** (1996) 389.

[21] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; **B431** (1994) 484.

[22] G. 't Hooft, Nucl. Phys. **B190** (1981) 455.

[23] Z. F. Ezawa and A. Iwazaki, Phys. Rev. **D25** (1982) 2681; **D26** (1982) 631.

[24] M. Stone and P. R. Thomas, Phys. Rev. Lett.**41** (1978) 351.
P. R. Carlo and M. Stone, Nucl. Phys. **B144** (1978) 513.

[25] H. Ichie, A. Tanaka and H. Suganuma, Nucl. Phys. **B** (Proc. Suppl.) **63A-C** (1998) 468.
H. Ichie, H. Suganuma and A. Tanaka Nucl. Phys. **A629** (1998) 82c.

[26] H. Ichie and H. Suganuma, Proc. of Int. Symp. on "Innovative Computational Methods in Nuclear Many-Body Problems", Osaka, Nov. 1997, (World Scientific) in press; hep-lat/9802032.

[27] J. M. Kosterlitz and D. J. Thouless, J. Phys. **C6** (1973) 1181.

[28] S. Maedan and T. Suzuki, Prog. Theor. Phys. **81** (1989) 229.
T. Suzuki, Prog. Theor. Phys. **80** (1988) 929; **81** (1989) 752.
S. Maedan, Y. Matsubara and T. Suzuki, Prog. Theor. Phys. **84** (1990) 130.

[29] H. Suganuma, S. Sasaki and H. Toki, Nucl. Phys. **B435** (1995) 207.
H. Suganuma, H. Toki, S. Sasaki and H. Ichie, Prog. Theor. Phys. (Suppl.) **120** (1995) 57.

[30] S. Sasaki, H. Suganuma and H. Toki, Prog. Theor. Phys. **94** (1995) 373; Phys. Lett. **B387** (1996) 145.

[31] S. Umisedo, H. Suganuma and H. Toki, Phys. Rev. **D57** (1998) 1605.

[32] H. Ichie, H. Suganuma and H. Toki, Phys. Rev. **D54** (1996) 3382; **D52** (1995) 2994.
H. Monden, H. Ichie, H. Suganuma and H. Toki Phys. Rev. **C57** (1998) 2564.

[33] K.-I. Kondo, Phys. Rev. **D57** (1998) 7467.

[34] A. Tanaka and H. Suganuma, Proc. of Int. Symp. on "Innovative Computational Methods in Nuclear Many-Body Problems", Osaka, Nov. 1997, (World Scientific) in press; hep-lat/9712027.

[35] T. Suzuki and I. Yotsuyanagi, Phys. Rev. **D42** (1990) 4257.

[36] S. Hioki, S. Kitahara, S. Kiura, Y. Matsubara, O. Miyamura, S. Ohno and T. Suzuki, Phys. Lett. **B272** (1991) 326.

[37] G. S. Bali, V. Bornyakov, M. Muller-Preussker and K. Schilling, Phys. Rev. **D54** (1996) 2863.

[38] O. Miyamura, Phys. Lett. **B353** (1995) 91; Nucl. Phys. **B** (Proc. Suppl.) **42** (1995) 538.
O. Miyamura and S. Origuchi, Proc. of Int. Workshop on "Color Confinement and Hadrons", (World Scientific, 1995) 65.

[39] R. M. Woloshyn, Phys. Rev. **D51** (1995) 6411.
F. X. Lee, R. M. Woloshyn and H. D. Trottier, Phys. Rev. **D53** (1996) 1532.

[40] A. S. Kronfeld, D. Schierholz and U.-J. Wiese, Nucl. Phys. **B293** (1987) 461.
A. S. Kronfeld, M. L. Laursen, G. Schierholz and U.-J. Wiese, Phys. Lett. **198B** (1987) 516.

[41] F. Brandstater, U.-J. Wiese and G. Schierholz, Phys. Lett. **B272** (1991) 319.

[42] A. Di Giacomo and G. Paffuti, Phys. Rev. **D56** (1997) 6816.

[43] K.-I. Kondo, CHIBA-EP-106, hep-th/9805153.

[44] A. Di Giacomo, Nucl. Phys. **B** (Proc. Suppl.) **47** (1996) 136 and references therein.

[45] M. I. Polikarpov, Nucl. Phys. **B** (Proc. Suppl.) **53** (1997) 134 and references therein.

[46] H. Suganuma, M. Fukushima, H. Ichie and A. Tanaka, Nucl. Phys. **B** (Proc. Suppl.) **65** (1998) 29.

[47] H. Suganuma, K. Itakura and H. Toki, hep-th/9512141.
H. Suganuma, K. Itakura, H. Toki and O. Miyamura, proc. of Int. Workshop on "Non-Perturbative Approaches to Quantum Chromodynamics", Trento, Italy, 1995, (PNPI press, 1995) 224, hep-ph/9512347.

[48] K. Amemiya and H. Suganuma, Proc. of Int. Symp. on "Innovative Computational Methods in Nuclear Many-Body Problems", Osaka, Nov. 1997, (World Scientific) in press; hep-lat/9712028.

[49] H. Suganuma, H. Ichie, A. Tanaka and K. Amemiya, Prog. Theor. Phys. (Suppl.) (1998) in press; hep-lat/9804027, and references therein.

[50] G. I. Poulis, Phys. Rev. **D54** (1996) 6974.

[51] H. Ichie and H. Suganuma, Proc. of Int. Workshop on "Future Directions in Quark Nuclear Physics", Adelaide, Australia, Mar. 1998, (World Scientific) in press; hep-lat/9807006.

[52] H. Georgi, "Lie Algebras in Particle Physics", (Benjamin/Cummings, 1982) 1.

FIGURE CAPTIONS

Fig.1: The gauge transformation property of Φ and gauge function $\Omega \in G/H$. The abelian gauge fixing is realized by $\Omega \in G/H$ so as to diagonalize Φ. (a) After the gauge transformation by $^\forall g \in G$, the operator Φ^g is diagonalized by the gauge function $\Omega^g = h[g]\Omega g^\dagger \in G/H$. (b) The gauge transformation property of the operator O^Ω defined in the abelian gauge. If O^Ω is H-invariant, O^Ω is found to be invariant under the whole gauge transformation of G.

Fig.2: (a) The probability distribution $P(R_{\rm Abel})$ of the abelian projection rate $R_{\rm Abel}$ at $\beta = 2.4$ on the 16^4 lattice from 40 gauge configurations. The solid curve denotes $P(R_{\rm Abel})$ in the MA gauge, and the dashed line denotes $P(R_{\rm Abel})$ without gauge fixing. (b) The average of the abelian projection rate $\langle R_{\rm Abel}\rangle$ in the MA gauge as the function of β. For comparison, we plot also $\langle R_{\rm Abel}\rangle$ without gauge fixing.

Fig.3: The off-diagonal gluon contribution on the Wilson loop, $W^{\rm off} \equiv \frac{\langle W_C[U_\mu(s)]\rangle}{\langle W_C[u_\mu(s)]\rangle}$, as the function of the perimeter length $L \equiv 2(I+J)$ in the MA gauge on 16^4 lattice with $\beta = 2.4$. The thick line denotes the theoretical estimation in Eq.(40) with the microscopic input $\langle \ln\{\cos\theta_\mu(s)\}\rangle_{\rm MA} \simeq -0.082$ at $\beta = 2.4$. The data of the Wilson loop with $I = 1$ or $J = 1$ are distinguished by the circle.

Fig.4: The inter-quark potential $V(r)$ as the function of the inter-quark distance r. The lattice data are obtained from the Wilson loop in the MA gauge on 16^4 lattice with $\beta = 2.4$ and $T = 7$. The square, the circle and the rhombus denote the full SU(2), the abelian and the off-diagonal contribution of the static potential, respectively. The thin line denotes the theoretical estimation in Eq.(43). Here, the lattice spacing a is determined so as to produce $\sigma = 0.89$ GeV/fm. Due to the artificial finite-size effect of the Wilson loop, the off-diagonal contribution $V^{\rm off}$ gets a slight linear part.

Fig.5: (a) The probability distributions $P(\chi_\mu)$ (solid line) and $P(\theta^3_\mu)$ (dash-dotted curve) in the MA gauge with the $\mathrm{U}(1)_3$ Landau gauge at $\beta = 2.4$ on the 16^4 lattice from 40 gauge configurations. (b) The probability distribution $P(\Delta\chi)$ of the correlation $\Delta\chi \equiv {\rm mod}_\pi(|\chi_\mu(s) - \chi_\mu(s+\hat\nu)|)$ in the same gauge at $\beta = 0$ (thin line), 1.0 (dotted curve), 2.4 (solid curve), 3.0 (dashed curve).

FIGURES

Fig.1

$$
\begin{tikzcd}
\Phi \arrow[r, "g \,\in\, G"] \arrow[d, "\Omega \,\in\, G/H"] & \Phi^{g} \arrow[dl, "\Omega^{g} \,\equiv\, h[g]\Omega g^{-1} \,\in\, G/H"] \\
\Phi_{\mathrm{diag}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
O \arrow[r, "g \,\in\, G"] \arrow[d, "\Omega"] & O^{g} \arrow[d, "\Omega^{g} \,\equiv\, h[g]\Omega g^{-1}"] \\
O^{\Omega} \arrow[r, "h[g] \,\in\, H"] & (O^{\Omega})^{h[g]}
\end{tikzcd}
$$

Fig.2

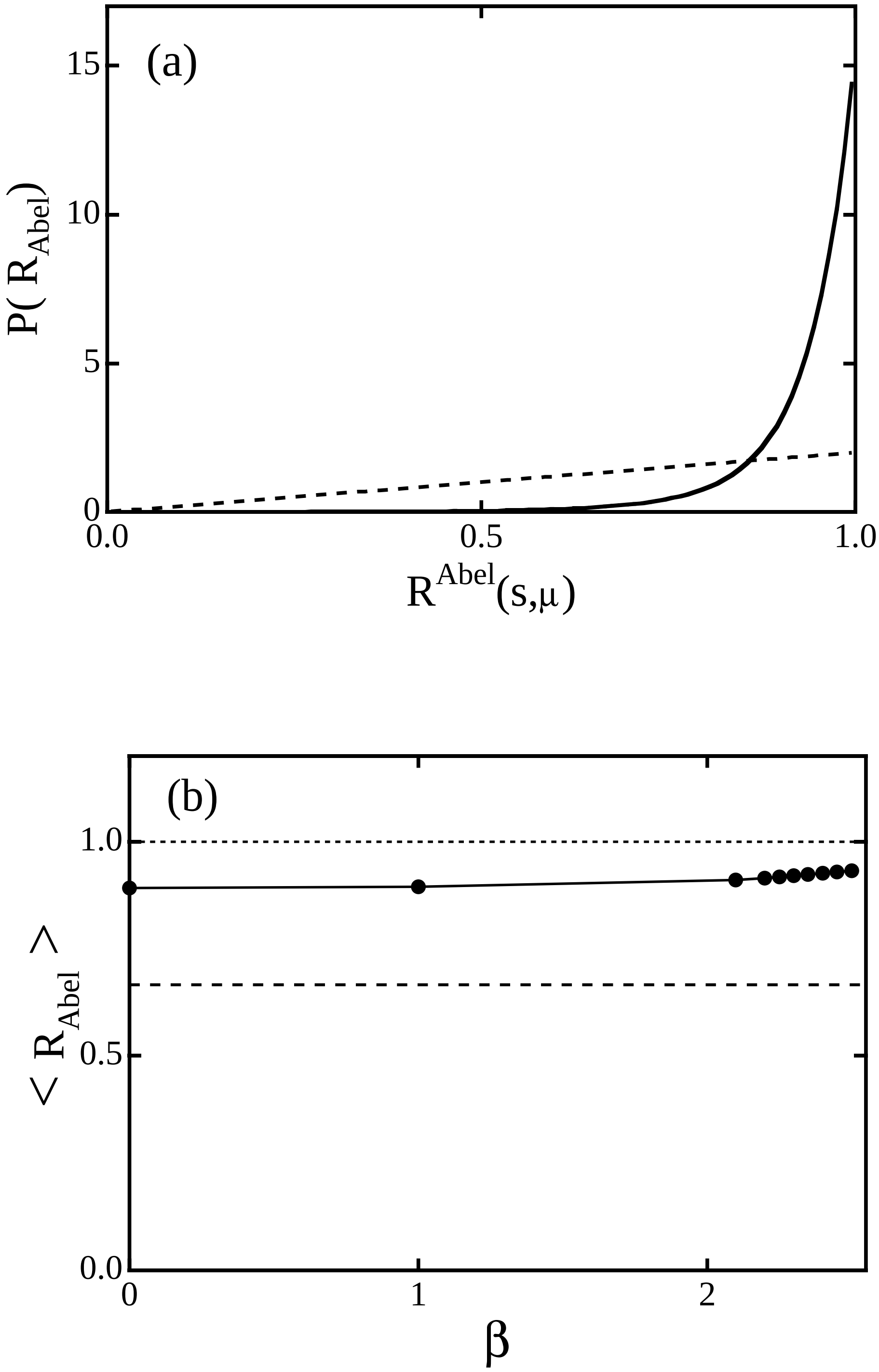

Fig.3

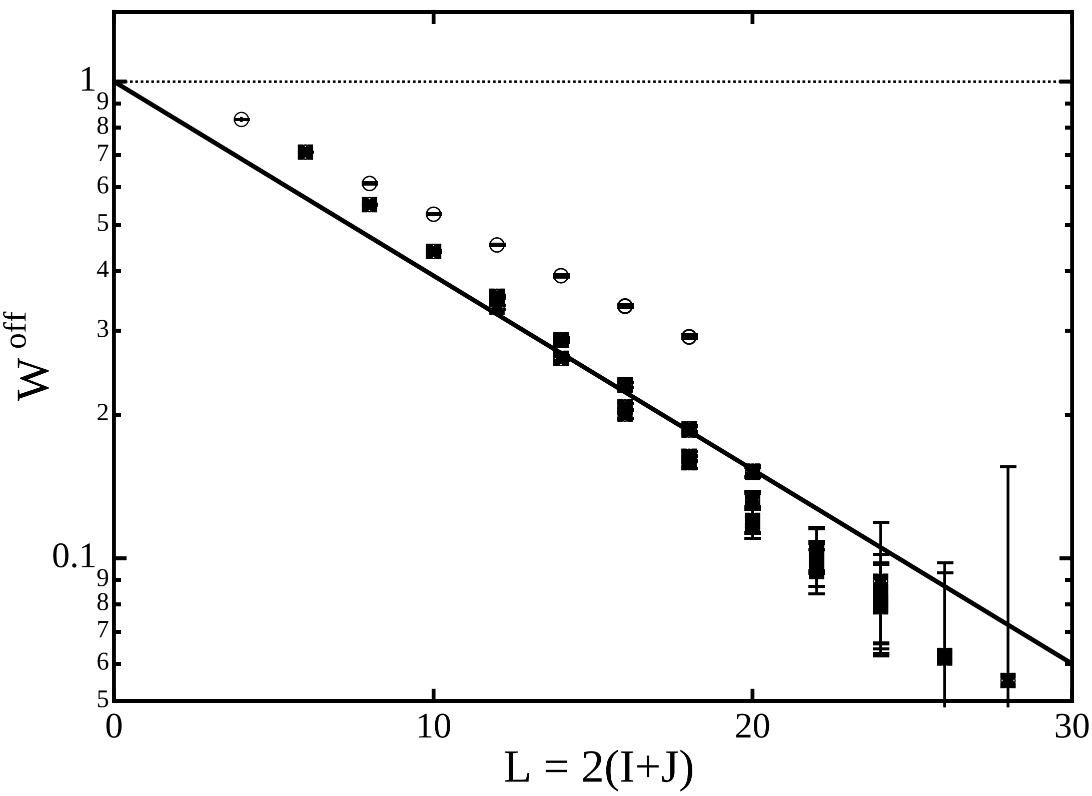

Fig.4

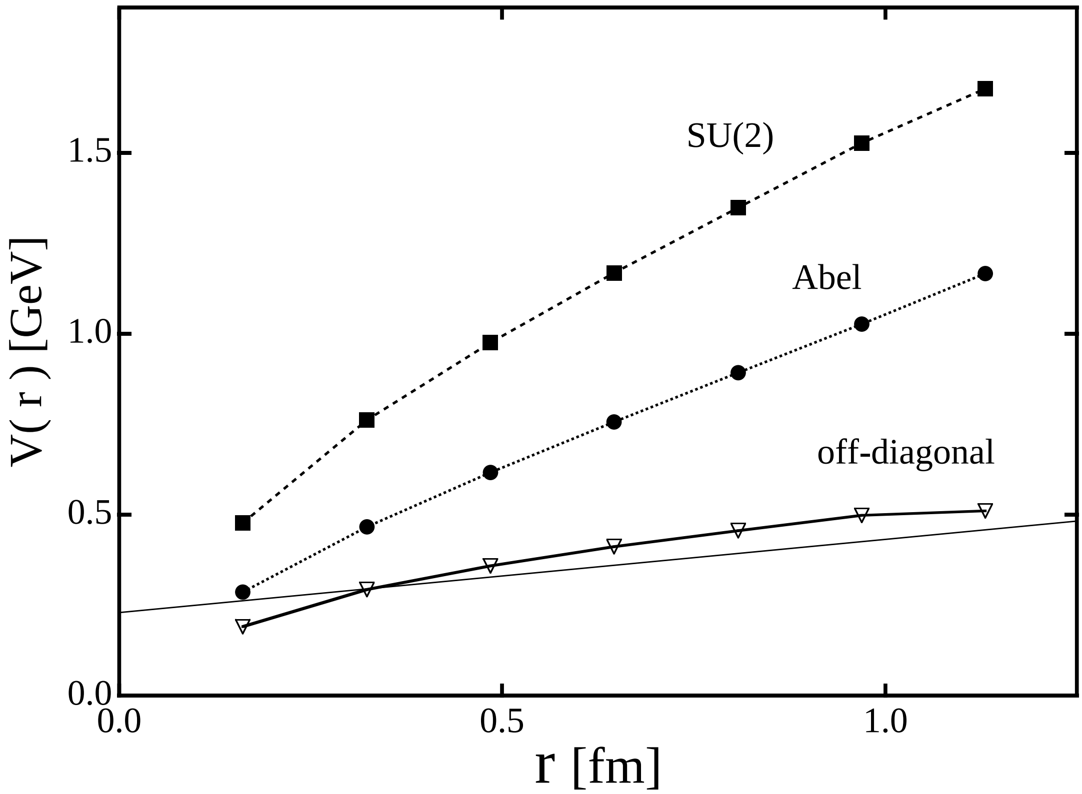

Fig.5